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ON A GENERALIZATION OF THE SET OF ASSOCIATED MINIMUM SURFACES*

By Arthur Sullivan Gale

1. Introduction. The well known memoirs on minimum surfaces by Sophus Lie† are based on the theorem that a minimum surface is a surface of translation whose generators are its minimum lines. As the generators are therefore imaginary curves the problem is suggested of considering all surfaces of translation whose generators are imaginary; the set of surfaces considered in this note consists of such surfaces.

By a *surface of translation* is meant a surface whose equations have the form

$$S: \quad \begin{aligned} x &= A(u) + A_1(v), \\ y &= B(u) + B_1(v), \\ z &= C(u) + C_1(v), \end{aligned}$$

for such a surface may be generated by translating either of the curves

$$\Gamma: \quad \begin{aligned} x &= A(u), \\ y &= B(u), \\ z &= C(u), \end{aligned} \qquad \text{or} \qquad \Gamma_1: \quad \begin{aligned} x &= A_1(v), \\ y &= B_1(v), \\ z &= C_1(v), \end{aligned}$$

so that a point of that curve describes the other. Through each point of S there will pass two curves on S which are congruent and parallel‡ to Γ and Γ_1 respectively; the totality of these curves may be spoken of as the two sets of *generators* of the surface.

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* Presented to the American Mathematical Society, February 28, 1903.

† *Math. Ann.*, vol. 14 (1879), p. 331 and vol. 15 (1880), p. 465. In these papers Lie has developed the elements of the theory of surfaces of translation, and derived certain properties of minimum surfaces, which Darboux (*Théorie des surfaces*, particularly §§ 220–225, 234, 239) has clarified and amplified in his elegant treatment of that subject. Inasmuch as these properties do not depend on the minimum property, but are enjoyed by all surfaces of translation, it is to be regretted that Darboux did not develop them in connection with the sections dealing with surfaces of translation as such (§§ 81, 82, 84). The surfaces of translation have received but passing reference in other text books; *e. g.*, Niewenglowski, *Géométrie analytique*, vol. 3, p. 159; Bianchi, *Differentialgeometrie* p. 112; Scheffers, *Anwendung der Differential- und Integralrechnung auf Geometrie*, vol. 2, p. 188.

‡ Two congruent curves in space may be said to be *parallel*, if they may be brought into coincidence by a translation.

For some purposes it is convenient to regard S as the locus of the middle point of a chord whose extremities lie on the curves

$$\begin{aligned} x &= 2A(u), \\ y &= 2B(u), \\ z &= 2C(u), \end{aligned} \qquad \text{and} \qquad \begin{aligned} x &= 2A_1(v), \\ y &= 2B_1(v), \\ z &= 2C_1(v). \end{aligned}$$

In particular, this point of view shows how it is possible to generate a real surface by translating one imaginary curve along another; for the middle point of a line joining two conjugate imaginary points is real.

In what follows we shall confine ourselves to surfaces of translation whose generators are imaginary. Now the coordinates of an imaginary curve may be given in either one of two ways: 1°) by *complex functions* of a *real parameter*; 2°) by functions of a *complex parameter*.

The surfaces of translation generated by imaginary curves of the first kind are of little interest. For, if we have two curves

$$\begin{aligned} x &= f_1(u) + if_2(u), \\ y &= g_1(u) + ig_2(u), \\ z &= h_1(u) + ih_2(u), \end{aligned} \qquad \text{and} \qquad \begin{aligned} x &= f_3(v) + if_4(v), \\ y &= g_3(v) + ig_4(v), \\ z &= h_3(v) + ih_4(v), \end{aligned}$$

where the f_i, g_i and h_i are real functions of the real parameters u and v, the surface of translation generated by them can be real only when the conditions

$$\begin{aligned} f_2(u) + f_4(v) &= 0, \\ g_2(u) + g_4(v) &= 0, \\ \text{and} \qquad h_2(u) + h_4(v) &= 0, \end{aligned}$$

are satisfied identically. Unless the six functions involved in these conditions reduce to constants which satisfy the conditions, the real points of the surface will consist at most of the points of a real curve; and if that reduction take place the surface may be generated by the real curves

$$\begin{aligned} x &= f_1(u), \\ y &= g_1(u), \\ z &= h_1(u), \end{aligned} \qquad \text{and} \qquad \begin{aligned} x &= f_3(v), \\ y &= g_3(v), \\ z &= h_3(v). \end{aligned}$$

We shall, therefore, consider only those surfaces whose generators are given by functions of complex parameters u and v, and we suppose that these functions are analytic.

In discussing surfaces whose generators are imaginary, the first question is naturally concerning their reality. A sufficient condition that S be real when Γ and Γ_1 are imaginary is evidently that $A_1(v)$, $B_1(v)$, and $C_1(v)$ be the conjugate imaginary functions* of $A(u)$, $B(u)$, and $C(u)$ respectively. And if we set $u = a + i\beta$ and $v = a - i\beta$, we see that this condition is also necessary; for if S be real, the imaginary parts of $A_1(v)$, $B_1(v)$, and $C_1(v)$ must be the imaginary parts, with their signs changed, of $A(u)$, $B(u)$, and $C(u)$ respectively, and hence the real parts of these functions can differ only by additive constants.† Hence:

A surface of translation whose generators are imaginary curves given in terms of conjugate imaginary parameters is real when and only when the generators of one set are the conjugate imaginary curves of the generators of the other set.

From the standpoint of differential equations, the surfaces of translation are evidently those surfaces whose coordinates satisfy the equation

$$\frac{\partial^2\theta}{\partial u \partial v} = 0.$$

If we set $u = a + i\beta$ and $v = a - i\beta$, this equation becomes

$$\frac{\partial^2\theta}{\partial a^2} + \frac{\partial^2\theta}{\partial \beta^2} = 0.$$

Hence: *The analytic surfaces of translation whose generators are imaginary are identical with the surfaces whose coordinates satisfy Laplace's equation.*

Hence we may legitimately speak of such surfaces as *harmonic surfaces.* To write the equations of S in such a form that its coordinates satisfy Laplace's equation, we have only to set $u = a + i\beta$ and $v = a - i\beta$, when x,

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* Two functions, $A(u)$ and $A_1(v)$, are said to be conjugate imaginary functions when they take on conjugate imaginary values for conjugate imaginary values of their arguments. $A(u)$ and $A_1(v)$ are the *same* functions of their respective arguments only when $A(u)$ is real for real values of u; but if $A(u) = a_1(u) + ia_2(u)$ then $A_1(v) = a_1(v) - ia_2(v)$.

† *Cf.* Picard, *Traité d'analyse,* vol. 2, chap. 1, §6. The system S (in Picard's notation) is the condition that the function $u(x, y) + iv(x, y)$ of the argument $x + iy$ have a unique derivative; it is also, evidently, the condition that the function $u(x, y) - iv(x, y)$ of the argument $x - iy$ have a unique derivative, *i. e.*, be an analytic function of $x - iy$.

y, and z are given as the real parts of $2A(u)$, $2B(u)$, and $2C(u)$ respectively. Conversely, to write the equations of the surface

$$x = f_1(\alpha, \beta),$$
$$y = f_2(\alpha, \beta),$$
$$z = f_3(\alpha, \beta),$$

where
$$\frac{\partial^2 f_i}{\partial \alpha^2} + \frac{\partial^2 f_i}{\partial \beta^2} = 0, \qquad (i = 1, 2, 3),$$

as the equations of a surface of translation, we determine the three functions $g_i(\alpha,\beta)$, $(i = 1, 2, 3)$, so that

$$\tfrac{1}{2}[f_1(\alpha, \beta) + ig_1(\alpha, \beta)] = A(u),$$
$$\tfrac{1}{2}[f_2(\alpha, \beta) + ig_2(\alpha, \beta)] = B(u),$$

and
$$\tfrac{1}{2}[f_3(\alpha, \beta) + ig_3(\alpha, \beta)] = C(u),$$

where $u = \alpha + i\beta$; then the surface may be generated by translating the curve

$$x = A(u),$$
$$y = B(u),$$
$$z = C(u),$$

so that one of its points describes the conjugate imaginary curve.

The fundamental quantities of a surface of translation are:

$$E = \Sigma\left(\frac{\partial x}{\partial u}\right)^2 = \Sigma\Big(A'(u)\Big)^2,$$

$$F = \Sigma\,\frac{\partial x}{\partial u}\,\frac{\partial x}{\partial v} = \Sigma\, A'(u)A_1'(v),$$

$$G = \Sigma\left(\frac{\partial x}{\partial v}\right)^2 = \Sigma\Big(A_1'(v)\Big)^2,$$

$$D = \frac{1}{\delta}\begin{vmatrix} \frac{\partial^2 x}{\partial u^2} & \frac{\partial^2 y}{\partial u^2} & \frac{\partial^2 z}{\partial u^2} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} = \frac{1}{\delta}\begin{vmatrix} A''(u) & B''(u) & C''(u) \\ A'(u) & B'(u) & C'(u) \\ A_1'(v) & B_1'(v) & C_1'(v) \end{vmatrix},$$

$$D' = \frac{1}{\delta}\begin{vmatrix} \frac{\partial^2 x}{\partial u \partial v} & \frac{\partial^2 y}{\partial u \partial v} & \frac{\partial^2 z}{\partial u \partial v} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} = 0,$$

$$D'' = \frac{1}{\delta}\begin{vmatrix} \frac{\partial^2 x}{\partial v^2} & \frac{\partial^2 y}{\partial v^2} & \frac{\partial^2 z}{\partial v^2} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} = \frac{1}{\delta}\begin{vmatrix} A_1''(v) & B_1''(v) & C_1''(v) \\ A'(u) & B'(u) & C'(u) \\ A_1'(v) & B_1'(v) & C_1'(v) \end{vmatrix},$$

where $\delta^2 = EG - F^2$.

Since $D' = 0$, the total curvature reduces to

$$k = \frac{1}{r_1 r_2} = \frac{DD''}{\delta^2};$$

and when $A(u)$, $B(u)$, and $C(u)$ are the conjugate functions of $A_1(v)$, $B_1(v)$, and $C_1(v)$ respectively, F will be real, G will be the conjugate of E, and D'' will be the negative conjugate of D (as is seen by interchanging the second and third rows of the determinant defining D''). Hence:

*The total curvature of any harmonic surface is everywhere negative.**

Remark. An investigation of the elementary questions presenting themselves in connection with the fundamental quantities of a surface of translation leads, with the exception of well known theorems on minimum surfaces and the theorem stated below, to negative results only; *e. g.*, the only surface of translation whose generators are its asymptotic lines is the plane.

If the generators form an orthogonal system they are the lines of curvature of the surface.

For $F = 0$ is the condition for orthogonality and we always have $D' = 0$. An illustration is afforded by any cylinder, if the generators be the right lines on the cylinder and their orthogonal trajectories; and, further, this property characterizes the cylinders. For the tangents to the generators of one set along the points of a generator of the other set are all parallel, and hence that generator can be orthogonal to the generators of the first set only when it is a plane curve whose plane is normal to the tangents to the generators of the first set. This being true for any two generators of the second set, whose planes, from the definition of S, must be parallel, the generators of the first set must be straight lines; unless, indeed, the generators of the second set be themselves straight lines. Hence:

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* This theorem has been stated by Professor Maclay; see note at the end of the present article.

The totality of cylinders may be defined as those surfaces of translation whose generators are their lines of curvature.

2. Associated Harmonic Surfaces. In seeking all minimum surfaces applicable upon a given real minimum surface

$$S:\quad \begin{aligned} x &= A(u) + A_1(v), \\ y &= B(u) + B_1(v), \\ z &= C(u) + C_1(v), \end{aligned} \qquad (E = G = 0)$$

one is led to the singly infinite set of *associated*[*] surfaces

$$S_a:\quad \begin{aligned} x' &= e^{ia}A(u) + e^{-ia}A_1(v), \\ y' &= e^{ia}B(u) + e^{-ia}B_1(v), \\ z' &= e^{ia}C(u) + e^{-ia}C_1(v), \end{aligned}$$

where a is real; of these surfaces the adjoint surface of Bonnet, obtained by setting $a = \frac{\pi}{2}$,—viz:

$$S_0:\quad \begin{aligned} x_0 &= i[A(u) - A_1(v)], \\ y_0 &= i[B(u) - B_1(v)], \\ z_0 &= i[C(u) - C_1(v)],\text{—} \end{aligned}$$

is very intimately connected with S. In this section we shall consider the properties of S_a on the supposition that S is any harmonic surface;[†] the sur-

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[*] Darboux, *l. c.*, book III, chap. V.

[†] If the generators of S be *real* the set of surfaces

$$\begin{aligned} x &= e^{a}A(u) + e^{-a}A_1(v), \\ y &= e^{a}B(u) + e^{-a}B_1(v), \\ z &= e^{a}C(u) + e^{-a}C_1(v), \end{aligned}$$

will possess the same properties as the set under discussion.

faces S_a may still be termed the surfaces associated with S, and S_0 the adjoint surface.

If E_a, F_a etc., denote the fundamental quantities of S_a, we have, in terms of those of S:

$$\begin{aligned} E_a &= e^{2ia}E, & D_a &= e^{ia}D, \\ F_a &= F, & D'_a &= D' = 0, \\ G_a &= e^{-2ia}G, & D''_a &= e^{-ia}D''. \end{aligned}$$

The following theorems are proved at once by means of these relations.

Corresponding areas of associated harmonic surfaces are equal.

For if do and do_a be the elements of surface of S and S_a respectively, we have

$$do_a = (E_aG_a - F_a^2)du\,dv = (EG - F^2)du\,dv = do.$$

The angles between the generators at corresponding points of associated harmonic surfaces are equal.

For if ω_a denote the angle between two generators of S_a we have

$$\cos\omega_a = \frac{F_a}{\sqrt{E_aG_a}} = \frac{F}{\sqrt{EG}} = \cos\omega.$$

Associated harmonic surfaces are applicable upon each other when and only when they are minimum surfaces.

For the square of the element of arc on S_a,—

$$ds_a^2 = e^{ia}Edu^2 + 2F du\,dv + e^{-2ia}Gdv^2,\text{—}$$

is independent of a when and only when $E = G = 0$.

The total curvature of associated harmonic surfaces is the same at corresponding points.

For the total curvature is given by

$$k_a = \frac{D_aD''_a}{\delta_a^2} = \frac{DD''}{\delta^2} = k.$$

Associated harmonic surfaces have the same spherical representation.

For the fundamental quantities of the spherical representation,—viz:

$$e_a = -d_a = \frac{G_a D_a^2}{\delta_a^2} = \frac{GD^2}{\delta^2},$$

$$f_a = -d_a' = -\frac{F_a D_a D_a''}{\delta_a^2} = -\frac{FDD''}{\delta^2},$$

$$g_a = -d_a'' = \frac{E_a D_a''^2}{\delta_a^2} = \frac{ED''^2}{\delta^2},—$$

are independent of a.

The coordinates of S_a may be easily expressed in terms of the coordinates of S and those of the adjoint surface S_0. For

$$e^{\pm ia} = \cos a \pm i \sin a,$$

and hence*

$$x' = x \cos a + x_0 \sin a,$$
$$y' = y \cos a + y_0 \sin a,$$

and

$$z' = z \cos a + z_0 \sin a,$$

In view of this fact we see that, although the mean curvature of S_a cannot be expressed simply in terms of that of S, yet it may be expressed simply in terms of the mean curvatures of S and S_0, as follows:

$$h_a = \frac{e^{ia}ED'' + e^{-ia}GD}{\delta^2} = h \cos a + h_0 \sin a.$$

NOTE. In a paper presented to the American Mathematical Society on Feb. 22, 1902, Professor Maclay considered the following singly infinite set of harmonic surfaces:

$$(1) \qquad \begin{aligned} x' &= x + x_0 \tan a, \\ y' &= y + y_0 \tan a, \\ z' &= z + z_0 \tan a, \end{aligned}$$

where x, y, z, x_0, y_0, and z_0 are the coordinates of corresponding points of any harmonic surface and the adjoint surface. These surfaces are similar by pairs to those treated above, the ratio of similitude being $\cos a$. Although many of the theorems given above are analogous to those given by Professor Maclay, I had been led to consider those surfaces before hearing his paper. If the surface with which Professor Maclay starts be written as a surface of translation :

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* *Cf.* Darboux, *l. c.*, § 211.

$$\begin{aligned} x &= A(u) + A_1(v), \\ y &= B(u) + B_1(v), \\ z &= C(u) + C_1(v), \end{aligned}$$

then the family he considered may be written :

$$\begin{aligned} x' &=(1 + i \tan a)A(u) + (1 - i \tan a)A_1(v), \\ y' &=(1 + i \tan a)B(u) + (1 - i \tan a)B_1(v), \\ z' &=(1 + i \tan a)C(u) + (1 - i \tan a)C_1(v) . \end{aligned} \tag{2}$$

It seems to me that this set of surfaces may be more readily studied in the form (2) than in the form (1) ; for in the form (1) the fundamental quantities of any surface of the set are found in terms of those of two surfaces, while in the form (2) they may be expressed in terms of those of a single surface.

YALE UNIVERSITY,
AUGUST, 1902.